

# A NOTE ON THE HYPERCONTRACTIVITY OF THE POLYNOMIAL BOHNENBLUST–HILLE INEQUALITY

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ABSTRACT. For  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $m$  a positive integer, we remark that there is a constant  $C$  so that, for all  $r \in [1, \frac{2m}{m+1}]$ , the supremum of the ratio between the  $\ell_r$  norm of the coefficients of any nonzero  $m$ -homogeneous polynomial  $P : \ell_\infty^n(\mathbb{K}) \rightarrow \mathbb{K}$  and its supremum norm is dominated by  $C^m \cdot n^{\frac{m}{r} - \frac{m+1}{2}}$  and, moreover, we prove that the exponent  $\frac{m}{r} - \frac{m+1}{2}$  is optimal.

## 1. INTRODUCTION

If  $P : \ell_\infty^n(\mathbb{C}) \rightarrow \mathbb{C}$  is an  $m$ -homogeneous polynomial defined by

$$P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$$

and  $\mathbb{D}^n$  is the unit polydisc in  $\mathbb{C}^n$ , let

$$\|P\|_r := \left( \sum_{|\alpha|=m} |a_\alpha|^r \right)^{1/r} \quad \text{and} \quad \|P\|_\infty := \sup_{z \in \mathbb{D}^n} |P(z)|.$$

The Sidon constant  $S_1(m, n)$  (see [6, 12]) is the smallest constant satisfying the inequality

$$(1.1) \quad \|P\|_1 \leq S_1(m, n) \|P\|_\infty$$

for all  $m$ -homogeneous polynomials  $P : \ell_\infty^n(\mathbb{C}) \rightarrow \mathbb{C}$ . From [6, 12] (see also the references therein) we know that there is an absolute constant  $C_{\mathbb{C}} > 0$  such that

$$(1.2) \quad S_1(m, n) \leq C_{\mathbb{C}}^m \cdot n^{\frac{m-1}{2}}.$$

It is also known that this result is sharp (we refer to [6, 12] and the references therein).

The inequality (1.1) is related to the famous polynomial Bohnenblust–Hille inequality for complex scalars (see [2]). Since the polynomial Bohnenblust–Hille inequality is hypercontractive for both real and complex scalars (see [4, 6]), there exists a constant  $C_{\mathbb{K}, BH} > 1$  such that

$$(1.3) \quad \|P\|_{\frac{2m}{m+1}} \leq C_{\mathbb{K}, BH}^m \|P\|_\infty$$

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for all positive integers  $n$  and all  $m$ -homogeneous polynomials  $P : \ell_\infty^n(\mathbb{K}) \rightarrow \mathbb{K}$ , with  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

It is well-known that (1.2) is a corollary of (1.3). In fact, according to [5, (2.1)] there are constants  $C, C_{\mathbb{C}} > 0$  such that

$$\begin{aligned}
 (1.4) \quad \sum_{|\alpha|=m} |a_\alpha| &\leq \left( \sum_{|\alpha|=m} |1|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{2m}} \left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \\
 &\leq \left( C^m \left( 1 + \frac{n}{m} \right)^m \right)^{\frac{m-1}{2m}} C_{\mathbb{C},BH}^m \|P\|_\infty \\
 &\leq C_{\mathbb{C}}^m \cdot n^{\frac{m-1}{2}} \|P\|_\infty.
 \end{aligned}$$

For related recent results we refer to [11, 13, 16] and [15] for a panorama of the subject.

Both (1.2) and the polynomial (and multilinear) Bohnenblust–Hille inequalities were originally conceived for complex scalars; the reason is that these inequalities were motivated by problems arising over the complex scalar field. In the last years, however, the interest in the Bohnenblust–Hille inequality encompassed the case of real scalars, mainly due to its connections with Quantum Information Theory (see [8]).

In this note we remark that (1.2) and (1.3) are particular cases of a continuum family of sharp inequalities for both complex and real scalars:

**Theorem 1.** *Let  $r \in [1, \frac{2m}{m+1}]$ ,  $m, n$  be positive integers and  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . There is an universal constant  $L_{\mathbb{K}} > 0$  such that*

$$(1.5) \quad \|P\|_r \leq L_{\mathbb{K}}^m \cdot n^{\left(\frac{m}{r} - \frac{m+1}{2}\right)} \|P\|_\infty$$

for all  $m$ -homogeneous polynomials  $P : \ell_\infty^n(\mathbb{K}) \rightarrow \mathbb{K}$ . Moreover, the power  $\frac{m}{r} - \frac{m+1}{2}$  is optimal.

It is worth mentioning that, in general, the adaptation of asymptotic results involving homogeneous polynomials from the complex setting to real scalars is not a straightforward task. In fact, sometimes polynomials present a completely different behavior when we change the scalar field from  $\mathbb{C}$  to  $\mathbb{R}$  (we refer to [3, page 58] for an illustrative example of this fact).

## 2. THE PROOF

Let  $r \in [1, \frac{2m}{m+1}]$ . The proof of (1.5) for complex scalars is easily obtained by adapting the argument used in (1.4). For real scalars, according to [4], if  $P : \ell_\infty^n(\mathbb{R}) \rightarrow \mathbb{R}$  is an  $m$ -homogeneous polynomial, then

$$\|P_{\mathbb{C}}\|_\infty \leq 2^{m-1} \|P\|_\infty,$$

where  $P_{\mathbb{C}}$  is the complexification of  $P$ ; this result goes back to the Visser's paper [17]. So, we obtain (1.4) for real scalars. It is also simple to see that the constant  $L_{\mathbb{K}}$  can be chosen independent of  $m, r, n$ . Now let us prove the optimality of the

exponent  $\frac{m}{r} - \frac{m+1}{2}$ ; for this task let us suppose that the result holds for a power  $q < \frac{m}{r} - \frac{m+1}{2}$ .

For each  $m, n$ , let

$$P_{m,n} : \ell_\infty^n(\mathbb{K}) \rightarrow \mathbb{K}$$

$$P_{m,n}(w) = \sum_{|\alpha|=m} \varepsilon_\alpha w^\alpha$$

be the  $m$ -homogeneous Bernoulli polynomial satisfying the Kahane–Salem–Zygmund inequality (note that this inequality is also valid for real scalars, see [10]).

The proof follows the lines of [10, Theorem 10.2]; the essence of this argument can be traced back to Boas' classical paper [1]. We can suppose  $n > m$ . As in [10], we have

$$\sum_{|\alpha|=m} |\varepsilon_\alpha|^r = p(n) + \frac{1}{m!} \prod_{k=0}^{m-1} (n-k),$$

where  $p(n) > 0$  is a polynomial of degree  $m-1$ . If (1.5) was valid with the power  $q$ , then there would exist a constant  $C_{q,\mathbb{K}} > 0$  so that

$$\left( \sum_{|\alpha|=m} |\varepsilon_\alpha|^r \right)^{\frac{1}{r}} \leq C_{q,\mathbb{K}}^m \cdot n^q \|P_{m,n}\|_\infty$$

$$\leq C_{q,\mathbb{K}}^m \cdot n^q \cdot C_{KSZ} \cdot n^{(m+1)/2} \sqrt{\log m},$$

where  $C_{KSZ} > 0$  is the universal constant from the Kahane–Salem–Zygmund inequality. Hence

$$C_{q,\mathbb{K}}^m C_{KSZ} \geq \frac{1}{n^q \cdot n^{(m+1)/2} \sqrt{\log m}} \left( p(n) + \frac{1}{m!} \prod_{k=0}^{m-1} (n-k) \right)^{1/r}$$

for all  $n$ . Raising both sides to the power of  $r$  and letting  $n \rightarrow \infty$  we obtain

$$(C_{q,\mathbb{K}}^m C_{KSZ})^r \geq \lim_{n \rightarrow \infty} \left( \frac{p(n)}{n^{qr} \cdot n^{r(m+1)/2} (\sqrt{\log m})^r} + \frac{s(n)}{n^{qr} \cdot n^{r(m+1)/2} (\sqrt{\log m})^r} \right),$$

with

$$s(n) = \frac{1}{m!} \prod_{k=0}^{m-1} (n-k).$$

Since  $q < \frac{m}{r} - \frac{m+1}{2}$ , we have  $\deg s = m > qr + r(m+1)/2$  and thus the limit above is infinity, a contradiction.

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